Solutions of two Diophantine equations $3^x + 9^y = z^2$ and $13^x + 9^y = z^2$

Md. Al-Amin Khan[1], Abdur Rashid[2], M.A.Halim[3] and Md. Sharif Uddin[4]

[1]Department of Mathematics, Jahangirnagar University, Bangladesh.
E-mail: alaminkhan@juniv.edu
[2]Department of Mathematics, Jahangirnagar University, Bangladesh.
E-mail: arashedju@yahoo.com
[3]Department of Natural Sciences, Daffodil International University, Bangladesh.
E-mail: abdulhalim359ju@gmail.com
[4]Department of Mathematics, Jahangirnagar University, Bangladesh.
E-mail: msharifju@yahoo.com

Abstract This paper is focused on solutions of two Diophantine equations of the type $p^x + 9^y = z^2$, where p is an odd prime number. We show that the Diophantine equation $3^x + 9^y = z^2$, where x, y and z are non-negative integers, has infinitely many solutions but $13^x + 9^y = z^2$ has no non-negative integer solution.

Keywords: Exponential Diophantine equation, Integer solutions.

1 Introduction

In recent, there have been a lot of studies about the Diophantine equation of the type $a^x + b^y = c^z$. In 2012, B. Sroysang [11] proved that $(1,0,2)$ is a unique solution (x, y, z) for the Diophantine equation $3^x + 5^y = z^2$ where x, y and z are non-negative integers. In 2013, B. Sroysang [12] showed that the Diophantine equation $3^x + 17^y = z^2$ has a unique non-negative integer solution $(x, y, z) = (1,0,2)$. In the same year, B. Sroysang [9] found all the solutions to the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. The solutions (x, y, z) are $(0,1,2)$, $(3,0,3)$ and $(4,2,5)$. In 2013, Rabago [8] showed that the solutions (x, y, z) of the two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$ where x, y and z are non-negative integers are $\{(1,0,2), (4,1,10)\}$ and $\{(1,0,2), (2,1,10)\}$, respectively.

In literature, a good amount of research [1, 2, 3, 4, 5, 6, 7, 10] is available for solving different kind of Diophantine equations.

In the present paper, we study the two Diophantine equations $3^x + 9^y = z^2$ and $13^x + 9^y = z^2$ and also find all solutions in non-negative integers.

2 Main Results

Theorem 2.1: *The Diophantine equation* $p^x + 1 = z^2$, *where* p *is an odd prime number and* x, y, z *are non-negative integers, is solvable only for* $p = 3$. *The solution is* $(x, z, p) = (1,2,3)$.

Proof: Let x and z be non-negative integers such that $p^x + 1 = z^2$, where p be an odd prime number. If $x = 0$, then $z^2 = 2$. It is impossible. If $z = 0$, then $p^x = -1$, which is also impossible.

Now for $x, z > 0$,

$$p^x + 1 = z^2$$

$$\text{or} \quad p^x = z^2 - 1 = (z-1)(z+1)$$

Let $z + 1 = p^\xi$ and $z - 1 = p^\psi$, where $\psi < \xi$, $\psi + \xi = x$. Then,

$$p^\psi (p^{\xi-\psi} - 1) = 2$$

Thus, $p^\psi = 1 \Rightarrow p^\psi = p^0 \Rightarrow \psi = 0$ and $p^{\xi-\psi} - 1 = 2 \Rightarrow p^\xi = 3$, which is possible only for $p = 3$ and $\xi = 1$. So $x = \psi + \xi = 0 + 1 = 1$,

$z = p^{\xi} - 1 = 3^1 - 1 = 2$.

Therefore, $(x, z, p) = (1,2,3)$ is the solution of $p^x + 1 = z^2$.

This proves the theorem.

Corollary 2.2: *The Diophantine equation* $3^x + 1 = z^2$ *has exactly one non-negative integer solution* $(x, z) = (1,2)$.

Corollary 2.3: *The Diophantine equation* $13^x + 1 = z^2$ *has no non-negative integer solution.*

Theorem 2.4: *The Diophantine equation* $1 + 9^x = z^2$ *has no non-negative integer solution.*

Proof: Suppose x and z be non-negative integers such that $1 + 9^x = z^2$. For $x = 0$, we have $z^2 = 2$. It is impossible. Let $x \geq 1$. Then $1 + 9^x = z^2$ gives us $3^{2x} = (z-1)(z+1)$. Let $z + 1 = 3^{\Pi_1}$ and $z - 1 = 3^{\Pi_2}$, where $\Pi_2 < \Pi_1$, $\Pi_1 + \Pi_2 = 2x$.

Therefore,

$$3^{\Pi_2}(3^{\Pi_1 - \Pi_2} - 1) = 2$$

Thus, $3^{\Pi_2} = 1$ or $\Pi_2 = 0$ and $3^{\Pi_1 - \Pi_2} - 1 = 2$ or $\Pi_1 = 1$. So $2x = 1 \Rightarrow x = \frac{1}{2}$, which is not acceptable since x is a non-negative integer. This completes the proof.

Theorem 2.5: *The Diophantine equation* $3^x + 9^y = z^2$ *has an infinitely many solutions of the form* $(x, y, z) = (2m + 1, m, 2.3^m)$, *where* m *is any non-negative integer.*

Proof: Suppose x, y and z be non-negative integers such that $3^x + 9^y = z^2$. If $x = 0$, then we have $1 + 9^y = z^2$ which has no solution by theorem 2.4. When $y = 0$ then by corollary 2.2, we have $x = 1$ and $z = 2$. Therefore, $(1,0,2)$ is a solution to $3^x + 9^y = z^2$. If $z = 0$, then $3^x + 9^y = 0$, which is not possible for any non-negative integers x and y.

Now we consider the following remaining cases.

Case – 1: $x = 1$. If $x = 1$ then we have $3 + 9^y = z^2$ $\Rightarrow 3 = z^2 - (3^y)^2$ $\Rightarrow 3 = (z + 3^y)(z - 3^y)$.

If $(z + 3^y) = 1$ and $(z - 3^y) = 3$, then $2z = 4$ $\Rightarrow z = 2$ and $2 + 3^y = 1 \Rightarrow 3^y = -1$, which is not possible. On the other hand, if $(z + 3^y) = 3$ and $(z - 3^y) = 1$, then $2z = 4$ $\Rightarrow z = 2$ and $2 + 3^y = 3 \Rightarrow 3^y = 1$, so $y = 0$. That is, we have $(x, y, z) = (1,0,2)$ is a solution to $3^x + 9^y = z^2$.

Case – 2: $y = 1$. If $y = 1$, then $3^x + 9 = z^2$ $\Rightarrow 3^x = z^2 - 9 \Rightarrow 3^x = (z+3)(z-3)$. Let $3^{\xi} = z + 3$ and $3^{\eta} = z - 3$, where $\xi > \eta, \xi + \eta = x$. Then

$$3^{\eta}(3^{\xi - \eta} - 1) = 2.3$$

Thus,

$3^{\eta} = 3 \Rightarrow \eta = 1$ and $3^{\xi - 1} - 1 = 2 \Rightarrow 3^{\xi - 1} = 3 \Rightarrow \xi = 2$.

So, $x = 1 + 2 = 3$ and $z = 6$. That is, for $y = 1$, we have the solution $(x, y, z) = (3,1,6)$.

Case – 3: $z = 1$. If $z = 1$, then $3^x + 9^y = 1$ which is not possible for any non-negative integers x and y.

Case – 4: $x, y, z > 1$. Now

$$3^x + 9^y = z^2$$

$$\text{or} \quad 3^x = (z + 3^y)(z - 3^y)$$

Let $z + 3^y = 3^{\Pi_1}$ and $z - 3^y = 3^{\Pi_2}$, where $\Pi_2 < \Pi_1$,

$$\Pi_1 + \Pi_2 = x.$$

Then,

$$3^{\Pi_2}(3^{\Pi_1 - \Pi_2} - 1) = 2.3^y$$

Thus, $3^{\Pi_2} = 3^y$ or $\Pi_2 = y$ and $3^{\Pi_1 - y} - 1 = 2$ this gives us $\Pi_1 = y + 1$. Then, $z - 3^y = 3^y$ that is, $z = 2.3^y$ which is solvable only for if z is of the form 2.3^m, where $m > 1$ is any integer. Therefore,

$2.3^m = 2.3^y \Rightarrow y = m$, where $m > 1$ is any integer.

$\therefore x = \Pi_1 + \Pi_2 = y + y + 1 = 2y + 1 = 2m + 1$ and $z = 2.3^m$.

Hence, for $x, y, z > 1$, the solution of the equation $3^x + 9^y = z^2$ is of the form $(x, y, z) = (2m+1, m, 2.3^m)$ where m is a positive integer such that $m > 1$.

Thus, when x, y, z are non-negative integers, solutions of the Diophantine equation $3^x + 9^y = z^2$ are given by $(x, y, z) = (2m + 1, m, 2.3^m)$, where m is any non-negative integer.

This completes the proof of the theorem.

Theorem 2.6. The Diophantine equation $13^x + 9^y = z^2$ has no non-negative integer solution.

Proof: Suppose x, y and z are non-negative integers for which $13^x + 9^y = z^2$. If $x = 0$, we have $1 + 9^y = z^2$ which has no solution by theorem 2.4. For $y = 0$ we use corollary 2.3. If $z = 0$, then $13^x + 9^y = 0$ which is not possible for any non-negative integers x and y.

ISSN 2229-5518

Now we consider the following remaining cases.

Case – 1: $x=1$. If $x=1$, then $13+9^y=z^2$ or $13=(z+3^y)(z-3^y)$. We have two possibilities. If $z+3^y=13$ and $z-3^y=1$, it follows that $2z=14$ or $z=7$ and $3^y=6$, a contradiction. On the other hand, $z+3^y=1$ and $z-3^y=13$, it follows that $2z=14$ or $z=7$ and $3^y=-6$ which is impossible.

Case – 2: $y=1$. If $y=1$, then $13^x+9=z^2$ or $13^x=(z+3)(z-3)$. Let $z+3=13^{\xi}$ and $z-3=13^{\eta}$, where $\eta<\xi, \xi+\eta=x$. Then $13^{\xi}-13^{\eta}=2.3$ or $13^{\eta}(13^{\xi-\eta}-1)=2.3$. Thus, $13^{\eta}=1$ and $13^{\xi-\eta}-1=6$, then this implies that $\eta=0$ and $13^{\xi}=7$, a contradiction.

Case – 3: $z=1$. If $z=1$, then $13^x+9^y=1$ which is not possible for any non-negative integers x and y.

Case – 4: $x,y,z>1$. Now

$$13^x+9^y=z^2 \text{ or}$$

$13^x=(z+3^y)(z-3^y)$

Let $z+3^y=13^{\Pi_1}$ and $z-3^y=13^{\Pi_2}$, where $\Pi_2<\Pi_1, \Pi_1+\Pi_2=x$. So $13^{\Pi_1}-13^{\Pi_2}=2.3^y$ or $13^{\Pi_2}(13^{\Pi_1-\Pi_2}-1)=2.3^y$. Thus, $13^{\Pi_2}=1$ and $13^{\Pi_1-\Pi_2}-1=2.3^y$ then these imply that $\Pi_2=0$ and $13^{\Pi_1}-1=2.3^y$. Since $13\equiv 1 \pmod 4$, it follows that $13^{\Pi_1}\equiv 1 \pmod 4$ i.e., $13^{\Pi_1}-1\equiv 0 \pmod 4$. But we see that $2.3^y \not\equiv 0 \pmod 4$. This is impossible.

3 Conclusion

In this paper, we have shown that the Diophantine equation $3^x+9^y=z^2$ has an infinitely many solutions and all the solutions are given by $(x,y,z)=(2m+1, m, 2.3^m)$, where m is any non-negative integer. On the other hand, we have also found that the Diophantine equation $13^x+9^y=z^2$ has no non-negative integer solution.

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